An Introduction to Some Spaces of Interval Functions

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Abstract

The paper gives a brief account of the spaces of interval functions defined through the concepts of H-continuity, D-continuity and S-continuity. All three continuity concepts generalize the usual concept of continuity for real (point valued) functions. The properties of the functions in these new spaces are discussed and investigated, preserving essential properties of the usual continuous real functions being of primary interest. Various ways in which the spaces of H-continuous, D-continuous and S-continuous interval functions complement the spaces of continuous real functions are discussed.

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1 Introduction

The aim of this paper is to give an introduction to the spaces of interval functions which have emerged in connection with applications to real analysis, approximation theory and partial differential equations. These spaces are all based on extending the concept of continuity of real functions to interval functions. We denote by $\mathbb{A}(\Omega)$ the set of all functions defined on an open set $\Omega \subset \mathbb{R}^n$ with values which are finite or infinite closed real intervals, that is,

$$\mathbb{A}(\Omega) = \{f : \Omega \to \mathbb{I}\,\overline{\mathbb{R}}\},$$

where $\mathbb{I}\,\overline{\mathbb{R}} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\},\ \underline{a} \le \overline{a}\}$. Given an interval $a = [\underline{a}, \overline{a}] \in \mathbb{I}\,\overline{\mathbb{R}}$,

$$w(a) = \begin{cases} \overline{a} - \underline{a} & \text{if} \quad \underline{a}, \overline{a} \text{ finite}, \\ \infty & \text{if} \quad \underline{a} < \overline{a} = \infty \text{ or } \underline{a} = -\infty < \overline{a}, \\ 0 & \text{if} \quad \underline{a} = \overline{a} = \pm\infty, \end{cases}$$

is the width of a. An extended real interval a is called a proper interval if $w(a) > 0$ and degenerate or point interval if $w(a) = 0$. Identifying $a \in \overline{\mathbb{R}}$ with the degenerate interval $[a, a] \in \mathbb{I}\,\overline{\mathbb{R}}$, we consider $\overline{\mathbb{R}}$ as a subset of $\mathbb{I}\,\overline{\mathbb{R}}$. In this way $\mathbb{A}(\Omega)$ contains the set of extended real valued functions, namely,

$$\mathcal{A}(\Omega) = \{f : \Omega \to \overline{\mathbb{R}}\}.$$

The set of all continuous real point valued functions $C(\Omega)$ is a subset of $\mathcal{A}(\Omega)$, that is, we have the inclusions

$$C(\Omega) \subseteq \mathcal{A}(\Omega) \subseteq \mathbb{A}(\Omega).$$

For every $x \in \Omega$ the value of $f \in \mathbb{A}(\Omega)$ is an interval $[\underline{f}(x), \overline{f}(x)]$. Hence, the function f can be written in the form $f = [\underline{f}, \overline{f}]$, where $\underline{f}, \overline{f} \in \mathcal{A}(\Omega)$ and $\underline{f}(x) \leq \overline{f}(x),\ x \in \Omega$. A trivial and direct extension of the concept of continuity of real functions to interval functions is given in the next definition.

Definition 1 *A function $f = [\underline{f}, \overline{f}] \in \mathbb{A}(\Omega)$ is called continuous if the functions $\underline{f}$ and $\overline{f}$ are both continuous extended real valued functions.*

If the values of f are finite intervals then

$$f - \text{continuous} \iff \underline{f}, \overline{f} \in C(\Omega).$$

An important issue in interval analysis is the construction of enclosures. Let $\mathcal{F}$ be a set of continuous real function on Ω, i.e., $\mathcal{F} \subset C(\Omega)$. A function $\psi \in \mathbb{A}(\Omega)$ is called a continuous interval enclosure of $\mathcal{F}$ if ψ is continuous and

$$\phi(x) \in \psi(x),\ x \in \Omega,\ \phi \in \mathcal{F}.$$

A continuous interval enclosure ψ of F is called minimal if for any continuous interval function h we have

$$\phi(x) \in h(x),\ x \in \Omega,\ \phi \in \mathcal{F} \Longrightarrow \psi(x) \subseteq h(x),\ x \in \Omega. \tag{1}$$

Clearly if the minimal continuous interval enclosure exists it is given by the following function also called interval hull of $\mathcal{F}$:

$$\text{hull}(\mathcal{F})(x) = \bigcap_{\psi \in \hat{\mathcal{F}}} \psi(x),\ x \in \Omega, \tag{2}$$

where $\hat{\mathcal{F}}$ is the set of all continuous interval enclosures of $\mathcal{F}$. The following example shows that a continuous minimal interval enclosure of a set of continuous functions does not always exist, that is, the interval hull of a set of continuous functions is not necessarily a continuous functions.

Example 2 *Let us consider the following subset of $C(\mathbb{R})$*

$$\mathcal{F} = \{\phi_\lambda : \lambda > 0\}$$

where

$$\phi_\lambda(x) = \begin{cases} \dfrac{1 - e^{-\lambda x}}{1 + e^{-\lambda x}} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

The interval hull of $\mathcal{F}$

$$f(x) \quad = \quad \text{hull}(\mathcal{F})(x) = \begin{cases} [0, 1] & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

is not a continuous interval function indicating that the set $\mathcal{F}$ does not have a minimal continuous interval enclosure.

The fact that the sets of continuous functions do not always have minimal continuous interval enclosures is not surprising as it relates to the fact that the set of continuous real functions $C(\Omega)$ is not Dedeking order complete, that is, the supremum and the infimum of a bounded set of continuous functions do not always exist in $C(\Omega)$. The order on $C(\Omega)$ considered in this regard is the usual one induced by the order on $\mathbb{R}$ in a point-wise way. In the same way partial order for interval functions can be point-wise induced by a partial order on $\mathbb{I}\overline{\mathbb{R}}$. Note that a partial order which extends the total order on $\overline{\mathbb{R}}$ can be defined on $\mathbb{I}\overline{\mathbb{R}}$ in more than one way, see Appendix 2. However, it proves useful to consider on $\mathbb{I}\overline{\mathbb{R}}$ the partial order $\leq$ defined by

$$[\underline{a},\overline{a}] \leq [\underline{b},\overline{b}] \Longleftrightarrow \underline{a} \leq \underline{b},\ \overline{a} \leq \overline{b}. \tag{3}$$

The partial order induced in $\mathbb{A}(\Omega)$ by (3) in a point-wise way, i.e.,

$$f \leq g \Longleftrightarrow f(x) \leq g(x),\ x \in \Omega, \tag{4}$$

is an extension of the usual point-wise order on the set of extended real valued functions $\mathcal{A}(\Omega)$ and in particular on the set $C(\Omega)$. A partial order naturally related to interval spaces is the relation inclusion. Similarly to (4) it can be defined for interval functions in a point-wise way, that is,

$$f \subseteq g \Longleftrightarrow f(x) \subseteq g(x),\ x \in \Omega, \tag{5}$$

We will see in the sequel that suitable subsets of the set of all Hausdorff continuous (H-continuous) interval functions considered with the order (4) are Dedeking order completions of well known sets of continuous function, thus improving an earlier result by Dilworth [7]. We will also define the set of Dilworth continuous (D-continuous) interval functions which contains the set of Hausdorff continuous functions as well as all interval hulls of bounded sets of continuous functions. This set provides a Dedekind order completion of the set of continuous interval functions both with respect to the order relation $\leq$ given in (4) and the order relation inclusion ($\subseteq$) given in (5). The even larger set of S-continuous functions contains also the topological completion of $C(\Omega)$ with respect to the Hausdorff distance between functions as defined in [15]. The three concepts of continuity, namely, H-continuity, D-continuity, S-continuity, considered in this paper in addition to the continuity concept in Definition 1 all generalize the concept of continuity of real functions in the sense that if a real function f is H-continuous, or D-continuous, or S-continuous, then it is continuous. These concepts are defined through the lower and upper Baire operators and the graph completion operator. Since these operators are fundamental for the exposition we recall in the next section, that is, Section 2, their definitions and essential properties. The use of extended real intervals is partially motivated by the fact that the definitions of the Baire operators involve infimums and supremums which might not exists in the realm of the usual (finite) real intervals. However, most applications involve functions which are finite or nearly finite in the sense of the following definitions.

Definition 3 *A function $f \in \mathbb{A}(\Omega)$ is called finite if*

$$|f(x)| < \infty,\ x \in \Omega$$

Definition 4 *A function $f \in \mathbb{A}(\Omega)$ is called nearly finite if there exists an open and dense subset D of Ω such that*

$$|f(x)| < \infty, \; x \in D$$

The concepts of continuity for interval functions mentioned above as well as the respective spaces are discussed in Section 3. The spaces of Hausdorff continuous interval functions are particularly considered in Section 4 while Sections 5 and 6 deal with the space of D-continuous functions and the space of S-continuous functions, respectively. Various ways in which the considered spaces of interval functions complement the spaces of continuous functions are discussed in the Sections 7-9.

2 The Baire operators and the graph completion operator

For every $x \in \Omega$, $B_\delta(x)$ denotes the open δ-neighborhood of x in Ω, that is,

$$B_\delta(x) = \{y \in \Omega : ||x - y|| < \delta\}.$$

Let D be a dense subset of Ω. The pair of mappings $I(D, \Omega, \cdot)$, $S(D, \Omega, \cdot) : \mathbb{A}(D) \to \mathcal{A}(\Omega)$ defined by

$$\begin{aligned} I(D, \Omega, f)(x) &= \sup_{\delta>0} \inf\{z \in f(y) : y \in B_\delta(x) \cap D\}, x \in \Omega, && (6)\\ S(D, \Omega, f)(x) &= \inf_{\delta>0} \sup\{z \in f(y) : y \in B_\delta(x) \cap D\}, x \in \Omega, && (7) \end{aligned}$$

are called lower Baire and upper Baire operators, respectively. Clearly for every $f \in \mathbb{A}(D)$ we have

$$I(D, \Omega, f)(x) \leq f(x) \leq S(D, \Omega, f)(x), \; x \in \Omega.$$

Hence the mapping $F : \mathbb{A}(D) \to \mathbb{A}(\Omega)$, called a graph completion operator, where

$$F(D, \Omega, f)(x) = [I(D, \Omega, f)(x), S(D, \Omega, f)(x)], \; x \in \Omega, \; f \in \mathbb{A}(\Omega), \tag{8}$$

is well defined and we have the inclusion

$$f(x) \subseteq F(D, \Omega, f)(x), \; x \in \Omega. \tag{9}$$

The name of this operator is derived from the fact that considering the graphs of f and $F(D, \Omega, f)$ as subsets of the topological space $\Omega \times \overline{\mathbb{R}}$, the graph of $F(D, \Omega, f)$ is the minimal closed set which is a graph of interval function on Ω and contains the the graph of f, see [15]. In the case when $D = \Omega$ the sets D and Ω will be usually omitted from the operators' argument lists, that is,

$$I(f) = I(\Omega, \Omega, f) \; , \quad S(f) = S(\Omega, \Omega, f) \; , \quad F(f) = F(\Omega, \Omega, f)$$

The lower Baire operator, the upper Baire operator and the graph completion operator are closely connected with the order relations in the domains of their arguments and their ranges. The following monotonicity properties with respect to the order relations (4) and (5) follow immediately from the definitions of the operators.

1. The lower Baire operator, the upper Baire operator and the graph completion operator are all monotone increasing with respect to the their functional argument, that is, if D is a dense subset of Ω, for every two functions $f, g \in \mathbb{A}(D)$ we have

$$f(x) \le g(x),\ x \in D \implies \begin{cases} I(D,\Omega,f)(x) \le I(D,\Omega,g)(x),\ x \in \Omega \\ S(D,\Omega,f)(x) \le S(D,\Omega,g)(x),\ x \in \Omega \\ F(D,\Omega,f)(x) \le F(D,\Omega,g)(x),\ x \in \Omega \end{cases} \tag{10}$$

2. The graph completion operator is inclusion isotone with respect to the functional argument, that is, if $f, g \in \mathbb{A}(D)$, where D is dense in Ω, then

$$f(x) \subseteq g(x),\ x \in D \Longrightarrow F(D,\Omega,f)(x) \subseteq F(D,\Omega,g)(x),\ x \in \Omega. \tag{11}$$

3. The graph completion operator is inclusion isotone with respect to the set D in the sense that if D_1 and D_2 are dense subsets of Ω and $f \in \mathbb{A}(D_1 \cup D_2)$ then

$$D_1 \subseteq D_2 \Longrightarrow F(D_1,\Omega,f)(x) \subseteq F(D_2,\Omega,f)(x), x \in \Omega. \tag{12}$$

   This, in particular, means that for any dense subset D of Ω and $f \in \mathbb{A}(\Omega)$ we have

$$F(D,\Omega,f)(x) \subseteq F(f)(x), x \in \Omega. \tag{13}$$

Further details about the Baire operators and the graph completion operator are given in Appendix 1 in connection with the semi-continuous function. From the properties presented there it can be easily seen that all three operators are idempotent and it fact we have the following stronger property: If the sets D_1 and D_2 are both dense in Ω and $D_1 \subseteq D_2$ then

$$\begin{aligned} I(D_2,\Omega,\cdot) \circ I(D_1,\Omega,\cdot) &= I(D_1,\Omega,\cdot) \\ S(D_2,\Omega,\cdot) \circ S(D_1,\Omega,\cdot) &= S(D_1,\Omega,\cdot) \\ F(D_2,\Omega,\cdot) \circ F(D_1,\Omega,\cdot) &= F(D_1,\Omega,\cdot) \end{aligned} \tag{14}$$

3 Three concepts of continuity

Using the graph completion operator we define the following three concepts.

Definition 5 *A function $f \in \mathbb{A}(\Omega)$ is called S-continuous, if $F(f) = f$.*

Definition 6 *A function $f \in \mathbb{A}(\Omega)$ is called Dilworth continuous or shortly D-continuous if for every dense subset D of Ω we have*

$$F(D,\Omega,f) = f.$$

Definition 7 *A function $f \in \mathbb{A}(\Omega)$ is called Hausdorff continuous, or H-continuous, if for every function $g \in \mathbb{A}(\Omega)$ which satisfies the inclusion $g(x) \subseteq f(x)$, $x \in \Omega$, we have $F(g)(x) = f(x)$, $x \in \Omega$.*

The following theorem indicates that the H-continuity is the strongest concept among the three while the S-continuity is the weakest.

Theorem 8 *Let $f \in \mathbb{A}(\Omega)$. Then the following implications hold*

$$f \text{ is H-continuous} \Longrightarrow f \text{ is D-continuous} \Longrightarrow f \text{ is S-continuous} \tag{15}$$

Proof. Let f be H-continuous. From the inclusion

$$f(x) \subseteq f(x),\ x \in \Omega,$$

and the Definition 7 it follows that

$$F(f) = f\ ,$$

which means that f is S-continuous. Let D be any dense subset of Ω. Due to the property (13) of the graph completion operator we have

$$F(D, \Omega, f)(x) \subseteq F(f)(x) = f(x),\ x \in \Omega$$

Using the idempotence of the graph completion operator, see (14), and the minimality property in the definition of Hausdorff continuity we have

$$F(D, \Omega, f)(x) = F(F(D, \Omega, f)) = f(x),\ x \in \Omega,$$

which shows that function f is D-continuous.

The second implication in the theorem follows immediately from Definition 6 by taking $D = \Omega$. ■

We will use the notations:

$\mathbb{F}(\Omega)$ - the set of all S-continuous interval functions defined on Ω,

$\mathbb{G}(\Omega)$ - the set of all D-continuous interval functions defined on Ω,

$\mathbb{H}(\Omega)$ - the set of all H-continuous interval functions defined on Ω,

where the term interval functions means extended real interval valued functions, that is, the elements of $\mathbb{A}(\Omega)$. The following inclusions follow from (15)

$$\mathbb{H}(\Omega) \subseteq \mathbb{G}(\Omega) \subseteq \mathbb{F}(\Omega).$$

With every interval function f one can associate S-continuous, D-continuous and H-continuous functions as stated in the next theorem.

Theorem 9 *Let $f \in \mathbb{A}(\Omega)$. Then*

(i) for every dense subset D of Ω the function $F(D, \Omega, f)$ is S-continuous;

(ii) the function $G(f) = [I(S(I(f))), S(I(S(f)))]$ is D-continuous;

(iii) both functions $F(S(I(f)))$ and $F(I(S(f)))$ are H-continuous and

$$F(S(I(f))) \leq F(I(S(f)))\ .$$

Proof. (i) Follows immediately from the idempotence of the operator F, see (14).

(iii) Denote $g = F(S(I(f)))$. The function g can also be written in the form $g = [\underline{g}, \overline{g}]$, where

$$\begin{aligned} \underline{g} &= I(S(I(f))), & (16) \\ \overline{g} &= S(S(I(f))) = S(I(f)). & (17) \end{aligned}$$

Let us assume that the function $h = [\underline{h}, \overline{h}] \in \mathbb{A}(\Omega)$ satisfies the inclusion

$$h(x) \subseteq g(x), \ x \in \Omega,$$

or, equivalently,

$$I(S(I(f))) \le \underline{h} \le \overline{h} \le S(I(f)). \tag{18}$$

Using the idempotence, see (14), and the monotonicity, see (10), of the operators I and S we have

$$\underline{g} = I(S(I(f))) = I(I(S(I(f)))) \le I(\underline{h}) \le I(\overline{h}) \le I(S(I(f))) = \underline{g}$$

Therefore

$$\underline{g} = I(\underline{h}) \tag{19}$$

In a similar way using the same idempotence and monotonicity properties with the inequalities (18) we obtain

$$\begin{aligned} \overline{g} \ &= \ S(I(f)) \ = \ S(S(I(f))) \ \ge \ S(\overline{h}) \ \ge \ S(\underline{h}) \\ &\ge \ S(I(S(I(f)))) \ \ge \ S(I(I(I(f)))) \ = \ S(I(f)) \ = \ \overline{g} \end{aligned}$$

Hence

$$\overline{g} = S(\overline{h}) \tag{20}$$

From (19) and (20) it follows that

$$g = F(h).$$

Hence, the function $g = F(S(I(f))$ is H-continuous. The H-continuity of $F(I(S(f)))$ is proved in the same way. The inequality in the theorem follows also from the properties (14) and (10). We have

$$\begin{aligned} F(S(I(f))) \ &\le \ F(S(I(S(f)))) \ = \ [I(S(I(S(f)))), S(I(S(f)))] \\ &\le \ [I(S(S(S(f)))), S(I(S(f)))] \ = \ [I(S(f)), S(I(S(f)))] \ = \ F(I(S(f))) \end{aligned}$$

(ii) Let $f \in \mathbb{A}(\Omega)$ and let D be any dense subset of Ω. It follows from (iii), which is proved above, that the function

$$g = [\underline{g}, \overline{g}] = F(S(I(f))) = [I(S(I(f))), S(I(f))]$$

is H-continuous, and therefore D-continuous, see Theorem 8. Hence

$$F(D, \Omega, g) = g$$

which in particular means that

$$I(D, \Omega, \underline{g}) = \underline{g}.$$

Therefore

$$I(D,\Omega,I(S(I(f)))) = I(S(I(f))).$$

In the same way we prove that

$$S(D,\Omega,S(I(S(f)))) = S(I(S(f))).$$

Thus we have

$$\begin{aligned} F(D,\Omega,G(f)) &= [I(D,\Omega,I(S(I(f)))), S(D,\Omega,S(I(S(f))))] \\ &= [I(S(I(f))), S(I(S(f)))] \ = \ G(f), \end{aligned}$$

which shows that the function $G(f)$ is D-continuous. ■

The operator $G : \mathbb{A}(\Omega) \longrightarrow \mathbb{G}(\Omega)$ is a useful tool for studying the D-continuous interval functions. Similar to the graph completion operator it can be defined for functions in $\mathbb{A}(D)$ where D is a dense subset of Ω. However, since this will not be used in this paper we will consider G only on $\mathbb{A}(\Omega)$ as stated in the following definition.

Definition 10 *The operator* $G : \mathbb{A}(\Omega) \longrightarrow \mathbb{G}(\Omega)$ *given by*

$$G(f) = [I(S(I(f))), S(I(S(f)))]$$

is called normalizing operator.

Theorem 9 is illustrated by the following example.

Example 11 *Consider the function* $f \in \mathbb{A}(\mathbb{R})$ *given by*

$$f(x) = \begin{cases} [-1,1] & \text{if} \quad x \in \mathbb{Z} \\ 0 & \text{if} \quad x \in (-\infty,0)\backslash\mathbb{Z} \\ [0,1] & \text{if} \quad x \in (0,\infty)\backslash\mathbb{Z} \end{cases}$$

We have $F(f) = f$ *meaning that* f *is S-continuous.*
The D-continuous function $G(f)$ *is given by*

$$G(f)(x) = \begin{cases} 0 & \text{if} \quad x \in (-\infty,0) \\ [0,1] & \text{if} \quad x \in [0,\infty) \end{cases}$$

Finally, we have the H-continuous functions

$$\begin{aligned} F(S(I(f)))(x) &= 0,\ x \in \mathbb{R} \\ F(I(S(f)))(x) &= \begin{cases} 0 & \text{if} \quad x \in (-\infty,0) \\ [0,1] & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x \in (0,\infty) \end{cases} \end{aligned}$$

The concepts of continuity given in Definitions 5, 6 and 7 are strongly connected to the concepts of semi-continuity of real functions. We have the following characterization of the fixed points of the lower and the upper Baire operators, see Appendix 1:

$$I(f) = f \iff f \text{ is lower semi-continuous on } \Omega \tag{21}$$

$$S(f) = f \iff f \text{ is upper semi-continuous on } \Omega \tag{22}$$

Hence for an interval function $f = [\underline{f}, \overline{f}]$

$$f \text{ is S-continuous } \Longleftrightarrow \left\{ \begin{array}{l} \underline{f} \text{ is lower semi-continuous} \\ \\ \overline{f} \text{ is upper semi-continuous} \end{array} \right. \tag{23}$$

The name of the D-continuous interval functions is due to a similar characterization through the normal upper semi-continuous and normal lower semi-continuous functions, see Appendix 1. We have

$$f \text{ is D-continuous } \Longleftrightarrow \left\{ \begin{array}{l} \underline{f} \text{ is normal lower semi-continuous} \\ \\ \overline{f} \text{ is normal upper semi-continuous} \end{array} \right. \tag{24}$$

The minimality condition associated with the Hausdorff continuous functions can also be formulated in terms of semi-continuous functions, namely, if $f = [\underline{f}, \overline{f}]$ is S-continuous then f is H-continuous if and only if

$$\{\phi \in \mathcal{A}(\Omega) : \phi \text{ is semi-continuous, } \underline{f} \leq \phi \leq \overline{f}\} = \{\underline{f}, \overline{f}\} \tag{25}$$

All three concepts of continuity defined in this section can be considered as generalizations of the concept of continuity of real functions in the sense that the only real (point valued) functions contained in each one of the above sets are the continuous functions. This is formally stated in the next theorem.

Theorem 12 *If the function $f \in \mathcal{A}(\Omega)$ is S-continuous, D-continuous or H-continuous than it is continuous.*

Proof. In view of the implications in Theorem 8, it is enough to consider the case when f is S-continuous. If $f = [f, f] \in \mathcal{A}(\Omega)$ is S-continuous, then f is both upper semi-continuous and lower semi-continuous on Ω, see (23). Therefore f is continuous on Ω. ∎

Historical Remark. The three concepts of continuity of interval functions discussed in this section, namely, S-continuity, D-continuity and H-continuity, are linked with the concepts of semi-continuity of real functions, see (23), (24) and (25). The lower and upper semi-continuous functions have been well known at least since the beginning of the 20th century and are usually credited to Baire, see [4]. The normal upper semi-continuous functions were introduced in 1950 by Dilworth in connection with the order completion of the lattice of continuous functions, see [7]. The concepts of S-continuity and H-continuity are both due to Sendov, see [14], [15]. It is quite interesting that pairing a lower semi-continuous function $\underline{f}$ with an upper semi-continuous function $\overline{f}$, such that $\underline{f} \leq \overline{f}$ produces a completely new concept from both algebraic and topological points of view, namely, the concept of S-continuous interval functions. It is shown in [15] that the set of all S-continuous functions on a compact subset of $\mathbb{R}$ is a complete metric space with respect to the Hausdorff distance between their graphs and has the rare and particularly useful property of being completely bounded. Similarly to the concept of S-continuity, here we consider interval functions given by pairs of a normal lower semi-continuous function $\underline{f}$ and a normal upper semi-continuous function $\overline{f}$, such that $\underline{f} \leq \overline{f}$ are considered. In honor of the contribution of Dilworth to this development these functions are called Dilworth continuous, or shortly D-continuous.

4 Spaces of Hausdorff continuous interval functions

The H-continuous functions, representing the strongest kind of continuity among the three considered above, are also similar to the usual continuous real functions in that they assume point (degenerate interval) values on a dense subset of the domain Ω. This is obtained from a Baire category argument. It was shown in [1] that for every $f \in \mathbb{H}(\Omega)$ the set

$$W_f = \{x \in \Omega : w(f(x)) > 0\} \tag{26}$$

is of first Baire category. Since $\Omega \subseteq \mathbb{R}^n$ is open this implies that for every $f \in \mathbb{H}(\Omega)$ the set

$$D_f = \{x \in \Omega : w(f(x)) = 0\} = \Omega \backslash W_f \text{ is dense in } \Omega. \tag{27}$$

Using that a finite or countable union of sets of first Baire category is also a set of first Baire category we have that for every finite or countable set $\mathcal{F}$ of Hausdorff continuous functions the set

$$D_{\mathcal{F}} = \{x \in \Omega : w(f(x)) = 0,\ f \in \mathcal{F}\} = \Omega \backslash \left(\bigcup_{f \in \mathcal{F}} W_f \right) \text{ is dense in } \Omega. \tag{28}$$

The property (27) can also be used to characterize the H-continuous functions as stated in the following theorem.

Theorem 13 *If the interval function f is D-continuous and assumes point (degenerate interval) values on a dense subset D of Ω, that is,*

$$w(f(x)) = 0,\ x \in D,\ D \text{ – dense subset of } \Omega,$$

then f is H-continuous.

Proof. Assume that the function f is D-continuous and satisfies the condition in the theorem. Let $g \in \mathbb{A}(\Omega)$ be such that

$$g(x) \subseteq f(x),\ x \in \Omega.$$

Then we have

$$g(x) = f(x),\ x \in D.$$

Hence

$$F(g)(x) \subseteq F(f)(x) = f(x) = F(D, \Omega, f)(x) = F(D, \Omega, g)(x) \subseteq F(g)(x), x \in \Omega,$$

where for the last inclusion we use the property (13) of the operator F. The above inclusions indicate that $F(g) = f$ which means that f is H-continuous. ■

It may appear at first that the minimality condition in Definition 7 applies at each individual point x of Ω, thus, not involving neighborhoods. However, the graph completion operator F does appear in this condition. And this operator according to (8) and therefore (6) and (7) does certainly refer to neighborhoods of points in Ω, a situation typical, among others, for the concept of continuity. Hence the following property of the continuous functions is preserved, [1].

Theorem 14 *Let f, g be H-continuous on Ω and let D be a dense subset of Ω. Then*

$$\begin{array}{llcll} \text{a)} & f(x) & \leq & g(x),\ x \in D \implies f(x) & \leq\ g(x),\ x \in \Omega, \\ \text{b)} & f(x) & = & g(x),\ x \in D \implies f(x) & =\ g(x),\ x \in \Omega. \end{array}$$

The following two theorems represent essential links with the usual point valued continuous functions.

Theorem 15 *Let $f = [\underline{f}, \overline{f}]$ be an H-continuous function on Ω.*
a) If $\underline{f}$ or $\overline{f}$ is continuous at a point $a \in \Omega$ then $\underline{f}(a) = \overline{f}(a)$.
b) If $\underline{f}(a) = \overline{f}(a)$ for some $a \in \Omega$ then both $\underline{f}$ and $\overline{f}$ are continuous at a.

The proof is given in [1]

Theorem 16 *Let D be a dense subset of Ω. If $f \in C(D)$ then*

$$\begin{array}{ll} (i) & F(D, \Omega, f)(x) = f(x),\ x \in D, \\ (ii) & F(D, \Omega, f) \in \mathbb{H}(\Omega), \end{array}$$

Proof. (i) Using that f is continuous on D, for every $x \in D$ we have

$$F(D, \Omega, f)(x) = F(D, D, f)(x) = f(x)$$

(ii) Let $g \in \mathbb{A}(\Omega)$ satisfy the inclusion

$$g(x) \subseteq F(D, \Omega, f)(x),\ x \in \Omega.$$

Using the property (11) of the operator F and its idempotence, see (14), we have

$$g(x) \subseteq F(g)(x) \subseteq F(F(D, \Omega, f))(x) = F(D, \Omega, f)(x), x \in \Omega.$$

Therefore

$$g(x) = F(D, \Omega, f)(x) = f(x), x \in D$$

Hence

$$F(g)(x) \subseteq F(D, \Omega, f)(x) = F(D, \Omega, g)(x) \subseteq F(g)(x),\ x \in \Omega,$$

where for the last inclusion we used the property (13) of the operator F. From the above inclusions we have

$$F(g)(x) = F(D, \Omega, f)(x)\ ,$$

which shows that the function $F(D, \Omega, f)$ is H-continuous. ■

We state below one of the most amazing properties of the set $\mathbb{H}(\Omega)$, namely, its order completeness, as well as the Dedekind order completeness of some of its important subsets. What makes this property so significant is the fact that with very few exceptions the usual spaces in Real Analysis or Functional Analysis are neither order complete nor Dedekind order complete, see Appendix 2 for the definitions of order completeness and Dedekind order completeness.

Theorem 17 *The set $\mathbb{H}(\Omega)$ of all H-continuous interval functions is order complete, that is, for every subset $\mathcal{F}$ of $\mathbb{H}(\Omega)$ there exist $u, v \in \mathbb{H}(\Omega)$ such that $u = \sup \mathcal{F}$ and $v = \inf \mathcal{F}$.*

Proof. First we will construct $u = \sup \mathcal{F}$. Consider the function

$$g(x) = \sup\{I(f)(x) : f \in \mathcal{F}\},\ x \in \Omega.$$

The function g, being a supremum of lower semi-continuous functions, is also a lower semi-continuous function, see Theorem 36 in Appendix 1. Therefore, according to Theorem 9 we have that $u = F(S(g)) = F(S(I(g)))$ is H-continuous, that is $u \in \mathbb{H}(\Omega)$. We will prove that u is the supremum of $\mathcal{F}$. More precisely, we will show that

i) u is an upper bound of $\mathcal{F}$, i.e. $f(x) \le u(x),\ \ x \in \Omega,\ \ f \in \mathcal{F}$,

ii) u is the smallest upper bound of $\mathcal{F}$, that is, for any function $h \in \mathbb{H}(\Omega)$,

$$f(x) \le h(x),\ \ x \in \Omega,\ \ f \in \mathcal{F} \implies u(x) \le h(x),\ \ x \in \Omega.$$

Using the monotonicity of the operators S and F, for every $f \in \mathcal{F}$ we have

$$\begin{array}{rcccccl} f(x) & \le & S(f)(x) & = & S(I(f))(x) & \le & S(g)(x),\ x \in \Omega, \\ f(x) & = & F(f)(x) & \le & F(S(g))(x) & = & u(x),\ x \in \Omega, \end{array}$$

which means that u is an upper bound of the set $\mathcal{F}$.

Assume that function $h \in \mathbb{H}(\Omega)$ is such that

$$f(x) \le h(x),\ \ x \in \Omega,\ \ f \in \mathcal{F}.$$

We have

$$g(x) = \sup\{I(f)(x) : f \in \mathcal{F}\} \le h(x),\ x \in \Omega,$$

which implies $S(g) \le S(h)$. Hence $u = F(S(g)) \le F(S(h)) = h$. Therefore, $u = \sup \mathcal{F}$.

The existence of $v = \inf \mathcal{F} \in \mathbb{H}(\Omega)$ is shown in a similar way. ∎

The following theorem gives a useful representation of the infimum and supremum of a subset of $\mathbb{H}(\Omega)$ in terms of the point-wise infimum and supremum, respectively.

Theorem 18 *Let $\mathcal{F} \subseteq \mathbb{H}(\Omega)$ and let the functions $\varphi, \psi \in \mathcal{A}(\Omega)$ be defined by*

$$\varphi(x) = \inf\{z \in f(x) : f \in \mathcal{F}\}\ ,\ \ \psi(x) = \sup\{z \in f(x) : f \in \mathcal{F}\}\ ,\ \ x \in \Omega.$$

Then

$$\inf \mathcal{F} = F(I(\varphi)),\ \ \ \sup \mathcal{F} = F(S(\psi)).$$

Proof. We will prove that $\sup \mathcal{F} = F(S(\psi))$. The proof of $\inf \mathcal{F} = F(I(\varphi))$ can be done in a similar way. In the proof of Theorem 17 the supremum $u = \sup \mathcal{F}$ was constructed in the form $u = F(S(g))$ where

$$g(x) = \sup\{I(f)(x) : f \in \mathcal{F}\},\ x \in \Omega.$$

From the inequality

$$I(f) \le f \le u,\ f \in \mathcal{F},$$

it follows that

$$g \le \psi \le u.$$

Using the monotonicity of the operators S and F, see (10), from the above inequalities we have

$$u = F(S(g)) \le F(S(\psi)) \le F(S(u)) = u.$$

Hence $F(S(\psi)) = u = \sup \mathcal{F}$. ∎

Example 19 *Let $\Omega = \mathbb{R}^n$. Consider the set*

$$\mathcal{F} = \{f_\delta : \delta > 0\} \subseteq C(\Omega),$$

where

$$f_\delta(x) = \begin{cases} 1 - \delta^{-1}||x|| & \text{if } x \in B_\delta(0) \\ 0 & \text{otherwise} \end{cases}$$

The point-wise infimum of the set $\mathcal{F}$ is

$$\varphi(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

which is not an H-continuous function. The infimum of $\mathcal{F}$ in $\mathbb{H}(\mathbb{R}^n)$ is $u(x) = 0$, $x \in \mathbb{R}^n$. Clearly, $u = F(I(\varphi))$.

Example 20 *Let $\Omega = \mathbb{R}$. Consider the set $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ where*

$$f_n(x) = \begin{cases} x^{-2n-1} & \text{if } \quad x \neq 0 \\ [-\infty, \infty] & \text{if } \quad x = 0 \end{cases} .$$

The point-wise supremum of $\mathcal{F}$ is

$$\psi(x) = \begin{cases} 0 & \text{if } \quad x < -1 \\ x^{-1} & \text{if } \quad -1 \leq x < 0 \\ \infty & \text{if } \quad 0 \leq x < 1 \\ x^{-1} & \text{if } \quad x \geq 1 \end{cases} .$$

Hence

$$\sup \mathcal{F} = \begin{cases} 0 & \text{if } \quad x < -1 \\ [-1, 0] & \text{if } \quad x = -1 \\ x^{-1} & \text{if } \quad -1 < x < 0 \\ [-\infty, \infty] & \text{if } \quad x = 0 \\ \infty & \text{if } \quad 0 < x < 1 \\ [1, \infty] & \text{if } \quad x = 1 \\ x^{-1} & \text{if } \quad x > 1 \end{cases} .$$

Remark 21 *The Theorem 18 establishes a close connection between the supremum (infimum) in $\mathbb{H}(\Omega)$ with respect to the partial order (4) and the point-wise supremum (infimum). However, as the above examples show, these two functions are clearly not the same, that is, for a set $\mathcal{F} \subseteq \mathbb{H}(\Omega)$, in general,*

$$(\sup \mathcal{F})(x) = \left(\sup_{f \in \mathcal{F}} f \right)(x) \neq \sup_{f \in \mathcal{F}} (f(x)) \ , \ x \in \Omega \ .$$

The next theorem states the Dedekind order completeness of some important subsets of $\mathbb{H}(\Omega)$.

Theorem 22 *The following subsets of $\mathbb{H}(\Omega)$ are all Dedekind order complete:*

(i) the set $\mathbb{H}_{bd}(\Omega)$ of all bounded H-continuous functions;

(ii) The set $\mathbb{H}_{ft}(\Omega)$ of all finite H-continuous functions;

(iii) The set $\mathbb{H}_{nf}(\Omega)$ of all nearly finite H-continuous.

The proof of (i) and (ii) can be found in [1]. Unlike the concepts of boundedness or finitness associated with the sets $\mathbb{H}_{bd}(\Omega)$ and $\mathbb{H}_{ft}(\Omega)$ which are well known, the concept of a function being nearly finite is relatively recent. Further details and properties associated with this concept are presented in Appendix 3, where the proof of (iii) can also be found.

5 The space of Dilworth continuous interval functions

The D-continuous functions are closely linked to the normalizing operator G as shown already in Theorem 9. Hence we will first state some properties of this operator.

Theorem 23 *The normalizing operator G is*

(i)monotone increasing with respect to the partial order (4), that is, for every two functions $f, g \in \mathbb{A}(\Omega)$ we have

$$f(x) \leq g(x),\ x \in \Omega \implies G(f)(x) \leq G(g)(x),\ x \in \Omega;$$

(ii) inclusion isotone, that is, for every two functions $f, g \in \mathbb{A}(\Omega)$ we have

$$f(x) \subseteq g(x),\ x \in \Omega \implies G(f)(x) \subseteq G(g)(x),\ x \in \Omega;$$

(iii) idempotent, that is, for every $f \in \mathbb{A}(\Omega)$ we have

$$G(G(f)) = G(f)$$

Proof. The statements (i) and (ii) follow directly for the monotonicity of the operators I and S, see (10).

(iii) The normalizing operator can be represented in the form

$$G(f) = [(S \circ I)(f), (I \circ S)(f)],\ f \in \mathbb{A}(\Omega).$$

Therefore, to prove that G is idempotent it is enough to show that the compositions

$$I \circ S \ \text{ and } \ S \circ I \tag{29}$$

are both idempotent. This follows from the idempotence, see (14), of the operators I and S and their monotonicity, see (10). Indeed, for every $f \in \mathbb{A}(\Omega)$ we have

$$\begin{aligned} ((I \circ S) \circ (I \circ S))(f) &= I(S(I(S(f)))) \leq I(S(S(S(f)))) = I(S(f)) = (I \circ S)(f) \\ ((I \circ S) \circ (I \circ S))(f) &= I(S(I(S(f)))) \geq I(I(I(I(f)))) = I(S(f)) = (I \circ S)(f) \end{aligned}$$

Hence

$$(I \circ S) \circ (I \circ S) = (I \circ S)$$

The idempotence of $S \circ I$ is proved in the same way. ■

From the monotonicity of the operators I and S one can also easily see that, for every $f \in \mathbb{A}(\Omega)$

$$G(f)(x) \subseteq F(f)(x),\ x \in \Omega.$$

Furthermore, we have

$$G \circ F = F \circ G = G.$$

The next two theorems give necessary and sufficient conditions for a function to be D-continuous, the first one - in terms of H-continuity, the second one - in terms of the operator G.

Theorem 24 *A function $f = [\underline{f}, \overline{f}] \in \mathbb{A}(\Omega)$ is D-continuous if and only if the functions $F(\underline{f})$ and $F(\overline{f})$ are both H-continuous.*

Proof. Assume first that the functions

$$\begin{aligned} F(\underline{f}) &= [\underline{f}, S(\underline{f})] \\ F(\overline{f}) &= [I(\overline{f}), \overline{f}] \end{aligned}$$

are H-continuous. From Theorem 8 it follows that these functions are D-continuous as well. Let D be a dense subset of Ω. Using that $F(\underline{f})$ is D-continuous we obtain

$$I(D, \Omega, \underline{f}) = \underline{f},$$

while using that $F(\overline{f})$ is D-continuous we obtain

$$S(D, \Omega, \overline{f}) = \overline{f}$$

The above two equations put together give

$$F(D, \Omega, f) = f$$

which means that f is D-continuous.

Let now $f = [\underline{f}, \overline{f}] \in \mathbb{A}(\Omega)$ be D-continuous. We will show that $g = [\underline{g}, \overline{g}] = F(\underline{f})$ is H-continuous. Clearly $\underline{g} = \underline{f}$ and $\overline{g} = S(\underline{f})$. Let $\varepsilon > 0$ be fixed. Since the function

$$w(g)(x) = w(g(x)) = \overline{g}(x) - \underline{g}(x),\ x \in \Omega,$$

is upper semi-continuous, the set

$$W_{g,\varepsilon} = \{x \in \Omega : w(g(x)) \geq \varepsilon\}$$

is closed. We will show that this set is nowhere dense as well. Assume the opposite, that is, there exists an open subset P of Ω such that $W_{g,\varepsilon}$ is dense in P. Then using again the fact that the function $w(g)$ is upper semi-continuous, for every $x \in P$ we have

$$\begin{aligned} w(g)(x) &= S(w(g))(x) = \inf_{\delta>0} \sup\{w(g)(y) : y \in B_\delta(x)\} \\ &\geq \inf_{\delta>0} \sup\{w(g)(y) : y \in B_\delta(x) \cap W_{g,\varepsilon}\} \ \geq \ \varepsilon \end{aligned}$$

Therefore

$$S(\underline{f})(x) - \underline{f}(x) = \overline{g}(x) - \underline{g}(x) = w(g)(x) \geq \varepsilon,\ x \in P. \tag{30}$$

Let us fix $x \in P$. Since P is open there exists $\delta_0 > 0$ such that $B_{\delta_0}(x) \subseteq P$. Now, using also (30) we obtain

$$\begin{aligned} S(\underline{f})(x) &= \inf_{\delta>0} \sup\{\underline{f}(y) : y \in B_\delta(x)\} \\ &= \inf_{0<\delta<\delta_0} \sup\{\underline{f}(y) : y \in B_\delta(x)\} \\ &\leq \inf_{0<\delta<\delta_0} \sup\{S(\underline{f})(y) - \varepsilon : y \in B_\delta(x)\} \\ &= S(S(\underline{f})) - \varepsilon \; = \; S(\underline{f}) - \varepsilon. \end{aligned}$$

The obtained contradiction shows that the set $W_{g,\varepsilon}$ is nowhere dense. Then the set

$$W_g = \bigcup_{\varepsilon>0} W_{g,\varepsilon}$$

being a union of closed, nowhere dense sets is a set of first Baire category. This implies that its complement in Ω

$$D_g = \Omega \setminus W_g = \{x \in \Omega : S(\underline{f})(x) = \underline{f}(x)\}$$

is dense in Ω. From the D-continuity of f it follows that

$$I(D_g, \Omega, \underline{f}) = \underline{f}.$$

Hence

$$I(S(\underline{f})) \leq I(D_g, \Omega, S(\underline{f})) = I(D_g, \Omega, \underline{f}) = \underline{f}$$

Since the inequality $I(S(\underline{f})) \geq \underline{f}$ is obvious we have

$$I(S(\underline{f})) = \underline{f}$$

Therefore g can be represented in the form

$$g = F(\underline{f}) = [\underline{f}, S(\underline{f})] = [I(S(I(f))), S(I(f)) = F(S(I(f)))$$

and the H-continuity of g follows from Theorem 9.

The H-continuity of $F(\overline{f})$ is proved in the same way. ∎

Theorem 25 *A function $f \in \mathbb{A}(\Omega)$ is D-continuous if and only if*

$$G(f) = f$$

Proof. Let $f \in \mathbb{A}(\Omega)$ be such that $G(f) = f$. It follows from Theorem 9 that $G(f)$ is D-continuous. Then f is D-continuous as well.

Assume now that $f \in \mathbb{A}(\Omega)$ is D-continuous. It follows from Theorem 24 that the functions

$$\begin{aligned} F(\underline{f}) &= [\underline{f}, S(\underline{f})] \\ F(\overline{f}) &= [I(\overline{f}), \overline{f}] \end{aligned}$$

are H-continuous. The minimality condition (25) indicates that

$$\begin{aligned} I(S(\underline{f})) &= \underline{f}, \\ S(I(\overline{f})) &= \overline{f}. \end{aligned}$$

Thus $G(f) = f$. ■

The D-continuous functions are not 'thin' as the H-continuous functions, that is, they may assume interval values on open subsets of Ω and on the whole of Ω for that matter. However, they still retain the properties of the continuous and H-continuous functions stated in Theorem 14, this time also extended with the relation inclusion as formulated in the next theorem.

Theorem 26 *Let f, g be D-continuous on Ω and let D be a dense subset of Ω. Then*

$$\begin{array}{ll} \text{a)} & f(x) \le g(x),\ x \in D \Longrightarrow f(x) \le g(x),\ x \in \Omega, \\ \text{b)} & f(x) = g(x),\ x \in D \Longrightarrow f(x) = g(x),\ x \in \Omega. \\ \text{c)} & f(x) \subseteq g(x),\ x \in D \Longrightarrow f(x) \subseteq g(x),\ x \in \Omega. \end{array}$$

The proof follows directly from Definition 6.

The set of D-continuous functions has similar order completeness properties to the properties of the set of Hausdorff continuous functions stated in Theorems 17 and 22.

Theorem 27 *(i) The set of $\mathbb{G}(\Omega)$ all H-continuous functions is order complete with respect to the partial order (4)*

(ii) Each of the following sets is Dedekind order complete with respect to the partial order (4):

- $\mathbb{G}_{bd}(\Omega)$ *the set of all bounded D-continuous interval functions;*
- $\mathbb{G}_{ft}(\Omega)$ *the set of all finite D-continuous interval functions;*
- $\mathbb{G}_{nf}(\Omega)$ *the set of all nearly finite D-continuous interval functions.*

Note that the use of the relation inclusion (5) makes little sense for H-continuous function since, due to the minimality condition in Definition 7, two H-continuous function are compared with respect to (5) if and only if they are equal. The situation in the case of D-continuous functions is completely different since these functions may assume proper interval values on open subsets of Ω or on the whole of Ω. It is rather interesting that all sets considered in Theorem 27 are also Dedekind order complete with respect to the relation inclusion given in (5).

Theorem 28 *The sets $\mathbb{G}_{bd}(\Omega)$, $\mathbb{G}_{ft}(\Omega)$, $\mathbb{G}_{nf}(\Omega)$ and $\mathbb{G}(\Omega)$ are all Dedekind order complete with respect to order relation inclusion given in (5).*

The proofs of both Theorem 27 and Theorem 28 follow from Theorem 17 by using that every D-continuous function can be represented through two H-continuous functions as provided by Theorem 24.

6 The space of S-continuous interval functions

The space $\mathbb{F}(\Omega)$ is much wider then the set of D-continuous functions. However, as demonstrated in the preceding sections, it is a useful embedding structure for the discussed spaces. In particular, one may note that it contains the completed graphs of all point-wise infima and suprema of sets of continuous functions. The space of all finite S-continuous function, which we denote by $\mathbb{F}_{ft}(\Omega)$ is studied in [15] in case when Ω is a compact real interval. It was shown that this space has two very interesting properties when considered as a metric space with respect to the Hausdorff distance, namely,

- $\mathbb{F}_{ft}(\Omega)$ is a complete metric space
- the closed and bounded subsets of $\mathbb{F}_{ft}(\Omega)$ are compact.

These properties can also be obtained as a consequence of a more general theorem about the Hausdorff metric on the set of compact subsets of $\mathbb{R}^n$, see [16], Theorems 1.8.2 and 1.8.3. The second property is particularly interesting in view of the fact that it cannot be attributed to the usual spaces of functions considered in Real Analysis or Functional Analysis. Similar properties can be formulated for the case when Ω is an open subset of $\mathbb{R}^n$, this however is beyond the scope of this paper and will be discussed in a separate publication.

7 Application: Dedekind order completion of sets of continuous functions

Here we consider the set $C(\Omega)$ of all continuous real functions defined on Ω, that is,

$$C(\Omega) = \{f : X \to \mathbb{R},\ f\text{−continuous on } \Omega\}$$

with a partial order defined point-wise as in (4), this time the values of the functions being real numbers. Clearly $C(\Omega)$ is a subset of $\mathbb{H}(\Omega)$, see Theorem 12. Furthermore, since the order between intervals (3) is an extension of the order in $\mathbb{R}$, the order in $\mathbb{H}(\Omega)$ is an extension of the order in $C(\Omega)$. In this way $C(\Omega)$ is embedded in the order complete set $\mathbb{H}(\Omega)$. We will also consider the subset $C_{bd}(\Omega)$ of $C(\Omega)$ consisting of all bounded continuous functions, that is,

$$C_{bd}(\Omega) = \{f \in C(\Omega) : \exists M \in R : |f(x)| \leq M,\ x \in \Omega\}.$$

The fact that both $C_{bd}(\Omega)$ and $C(\Omega)$ are neither order complete nor Dedekind order complete is well known and can be shown by trivial examples, see Appendix 2 for the respective definitions. A general result on Dedekind order completion of partially ordered sets was established by MacNeilly in 1937, see [8] for a more recent presentation. The problem of order completion of $C(\Omega)$ is particularly addressed in [9]. In our approach we seek characterization of the Dedekind order completions of $C_{bd}(\Omega)$ and $C(\Omega)$ as subsets of $\mathbb{H}(\Omega)$. Obviously, since the set $\mathbb{H}(\Omega)$ is order complete it also contains the Dedekind order completions of these sets. An interesting aspect of this approach is that the Dedekind order completions of the mentioned sets of continuous functions are constructed again as sets of functions defined on the same domain Ω. An earlier result of Dilworth is of

similar nature. In [7] it is proved that the set of the normal upper semi-continuous real functions defined on Ω is a Dedekind order completion of $C_{bd}(\Omega)$. In [1] an *alternative* characterization of the Dedekind order completion of $C_{bd}(\Omega)$ was found and obtained again as a set of functions, this time given as a subset of the set $\mathbb{H}(\Omega)$. Furthermore, the Dedekind order completion of $C(\Omega)$ is also characterized as a subset of $\mathbb{H}(\Omega)$. More precisely, in terms of the notations adopted in this paper, we have

- $\mathbb{H}_{bd}(\Omega)$ is a Dedekind order completion of $C_{bd}(\Omega)$;
- $\mathbb{H}_{ft}(\Omega)$ is a Dedekind order completion of $C(\Omega)$.

Following the method used to prove the second statement above, see Theorem 10 in [1] one can show that the following representation of H-continuous functions.

Theorem 29 *Let $f = [\underline{f}, \overline{f}] \in \mathbb{H}(\Omega)$.*

(i) If $f(x) > -\infty$, $x \in \Omega$ then

$$f = \sup\{g \in C(\Omega) : g \leq f\}$$

(ii) If $f(x) < +\infty$, $x \in \Omega$ then

$$f = \inf\{g \in C(\Omega) : g \geq f\}$$

We should note that the representation of the H-continuous functions given in the above theorem essentially uses Dedekind cuts. Under the conditions considered in this theorem the point-wise representation given in Theorem 18 simplifies in the following way:

$$\text{case } (i): \quad \underline{f}(x) = \sup\{g(x) : g \in C(\Omega) : g \leq f\},\ x \in \Omega; \tag{31}$$

$$\text{case } (ii): \quad \overline{f}(x) = \inf\{g(x) : g \in C(\Omega) : g \leq f\},\ x \in \Omega; \tag{32}$$

Obviously, the Theorem 29 does not give representation in terms of functions in $C(\Omega)$ for all H-continuous functions since there are H-continuous functions with values involving both $-\infty$ and $+\infty$. However, using Theorem 29 it is not difficult to prove:

Theorem 30 *For every $f \in \mathbb{H}(\Omega)$ we have*

$$f = \inf\{\sup\{g \in C(\Omega) : g \leq \sup\{f, m\}\} : m \in \mathbb{R}\}$$

In the above theorem the letter m denotes both the real number m and the constant function with value m on Ω. This theorem shows that $\mathbb{H}(\Omega)$ is the minimal order complete set containing $C(X)$, that is,

> $\mathbb{H}(\Omega)$ is an order completion of $C(\Omega)$.

8 Application: Order isomorphic representation of piece-wise smooth functions

In this section we consider the space $C_{nd}(\Omega)$ of real (point) valued functions which are continuous every where on Ω except for a closed, nowhere dense subset $\Gamma \subseteq \Omega$. Obviously every possible kind of piece-wise continuous functions are included in the set $C_{nd}(\Omega)$ because the set Γ may have arbitrary shapes. In addition Γ may also have a positive Lebesgue measure. The space $C_{nd}(\Omega)$ is used, among others, with the order completion method for solution of nonlinear PDEs, see [13].

More precisely the set $C_{nd}(\Omega)$ is defined as

$$C_{nd}(\Omega) = \left\{ f \;\middle|\; \begin{array}{l} \exists \Gamma \subset \Omega \text{ closed, nowhere dense:} \\ \text{i) } f : \Omega \backslash \Gamma \longmapsto \mathbb{R} \\ \text{ii) } f \in C(\Omega \backslash \Gamma) \end{array} \right\} \tag{33}$$

Since the only point valued functions in the set $\mathbb{H}(\Omega)$ are the functions which are continuous on the whole of Ω, see Theorem 12, the set $C_{nd}(\Omega)$ is not a subset of $\mathbb{H}(\Omega)$. We shall define a mapping from $C_{nd}(\Omega)$ to $\mathbb{H}(\Omega)$ which gives are representation of the functions in $C_{nd}(\Omega)$ through Hausdorff continuous functions. To this end we proceed as follows.

Let $u \in C_{nd}(\Omega)$. According to (33), there exists a closed, nowhere dense set $\Gamma \subset \Omega$ such that $u \in C(\Omega \setminus \Gamma)$. Since $\Omega \setminus \Gamma$ is open and dense in Ω, we can define

$$F_0(u) = F(\Omega \setminus \Gamma, \Omega, u) \tag{34}$$

The closed, nowhere dense set Γ used in (34), is not unique. However, we can show that the value of $F(\Omega \setminus \Gamma, \Omega, u)$ does not depend on the set Γ in the sense that for every closed, nowhere dense set Γ such that $u \in C(\Omega \setminus \Gamma)$ the value of $F(\Omega \setminus \Gamma, \Omega, u)$ remains the same.

Let Γ_1 and Γ_2 be closed, nowhere dense sets such that $u \in C(\Omega \backslash \Gamma_1)$ and $u \in C(\Omega \backslash \Gamma_2)$. Then the set $\Gamma_1 \cup \Gamma_2$ is also closed and nowhere dense. According to Theorem 16 the functions $F(\Omega \setminus \Gamma_1, \Omega, u)$, $F(\Omega \setminus \Gamma_2, \Omega, u)$ and $F(\Omega \setminus (\Gamma_1 \cup \Gamma_2), \Omega, u)$ are all H-continuous and for every $x \in \Omega \setminus (\Gamma_1 \cup \Gamma_2)$ we have

$$F(\Omega \setminus \Gamma_1, \Omega, u)(x) = F(\Omega \setminus \Gamma_2, \Omega, u)(x) = F(\Omega \setminus (\Gamma_1 \cup \Gamma_2), \Omega, u)(x) = u(x).$$

Since $\Omega \setminus (\Gamma_1 \cup \Gamma_2)$ is dense in Ω, Theorem 14 implies that

$$F(\Omega \setminus \Gamma_1, \Omega, u) = F(\Omega \setminus \Gamma_2, \Omega, u) = F(\Omega \setminus (\Gamma_1 \cup \Gamma_2), \Omega, u).$$

Therefore, the mapping

$$F_0 : C_{nd}(\Omega) \longmapsto \mathbb{A}(\Omega)$$

is unambiguously defined through (34). In analogy with (8), we call F_0 a graph completion mapping on $C_{nd}(\Omega)$. As mentioned already above it follows from Theorem 16 that for every $u \in C_{nd}(\Omega)$ we have

$$F_0(u) \in \mathbb{H}(\Omega).$$

Furthermore, if $u \in C(\Omega \setminus \Gamma)$ we have

$$F_0(u)(x) = u(x), \; x \in \Omega \setminus \Gamma. \tag{35}$$

The above identity shows that the values of the function $F_0(u)$ are finite on the open and dense set $\Omega \setminus \Gamma$. Hence, $F_0(u)$ is nearly finite, see Definition 4. Thus, we have

$$F_0 : C_{nd}(\Omega) \longmapsto \mathbb{H}_{nf}(\Omega). \tag{36}$$

The following theorem shows that a function $f \in C_{nd}(\Omega)$ can be identified with $F_0(f)$ in a very direct way.

Theorem 31 *Let $f \in C_{nd}(\Omega)$.*

(i) If $f \in C(\Omega \setminus \Gamma)$ then

$$f(x) = F_0(f)(x),\ x \in \Omega \setminus \Gamma;$$

(ii) If $W_{F_0(f)}$ is the subset of Ω defined through (26) for the function $F_0(f)$ then

$$f(x) = F_0(f)(x),\ x \in \Omega \setminus W_{F_0(f)}.$$

The above theorem shows that

- the largest set on which $f \in C_{nd}(\Omega)$ can be defined in a continuous way is $\Omega \setminus W_{F_0(f)}$, that is, for every set Γ which is associated with f in terms of (33) we have $W_{F_0(f)} \subseteq \Gamma$;
- the set $W_{F_0(f)}$ is a closed, nowhere dense set and not merely a set of first Baire category.

Therefore, every function $f \in C_{nd}(\Omega)$ can be identified with $F_0(f)$ in the following way:

- If the set $\Gamma \subseteq \Omega$ is associated with f in terms of (33) then f is defined and continuous on the set
$$\Omega \setminus \Gamma \subseteq \Omega \setminus W_{F_0(f)}$$
- f can be produced in a continuous way on the open and dense set $\Omega \setminus W_{F_0(f)}$ and it is identical with $F_0(f)$ on this set;
- f can not be produced in a continuous way of a subset of Ω which is larger than $W_{F_0(f)}$.

Further research is aimed at applying the results discussed in this section to improve the regularity results associated with the order completion method in [13].

9 Application: The set of D-continuous functions contains all interval hulls of subsets of $C(\Omega)$

It was shown in Example 2 that the interval hull of a set of continuous functions is not always a continuous function. On the other side, from the representation (2) it is easy to see that the graph of the hull is a closed subset of $\Omega \times \overline{\mathbb{R}}$ implying that the interval hull is an S-continuous function. The following theorem gives more precise characterization.

Theorem 32 *For every set of continuous real functions $\mathcal{F} \subseteq C(\Omega)$ the interval hull defined through (2) is a D-continuous interval function.*

Proof. For simplicity we will present the proof only in the case when the set $\mathcal{F}$ has a finite continuous enclosure.

Let $g = [\underline{g}, \overline{g}] = \mathrm{hull}(\mathcal{F})$. Denote

$$\mathcal{F}_1 = \{\varphi \in C(\Omega) : \varphi \leq f,\ \forall f \in \mathcal{F}\}$$

Consider $\mathcal{F}$ as a subset of the order complete set $\mathbb{H}(\Omega)$ and let $\inf \mathcal{F}$ and $\sup \mathcal{F}$ be the infimum and the supremum of $\mathcal{F}$, respectively. Then

$$\mathcal{F}_1 = \{\varphi \in C(\Omega) : \varphi \leq \inf \mathcal{F}\} \tag{37}$$

From Theorem 29(i) we obtain

$$\inf \mathcal{F} = \sup \mathcal{F}_1 \tag{38}$$

It follows from (2) that $\underline{g}$ can be represented as

$$\underline{g}(x) = \sup\{\varphi(x) : \varphi \in \mathcal{F}_1\},\ x \in \Omega. \tag{39}$$

Then using Theorem 18 we have

$$\sup \mathcal{F}_1 = F(S(\underline{g}))$$

Hence

$$\inf \mathcal{F} = F(S(\underline{g})) = [I(S(\underline{g})), S(\underline{g})]$$

Now using (31) we obtain

$$\begin{aligned} I(S(\underline{g}))(x) &= \sup\{\varphi(x) : \varphi \in C(\Omega) : \varphi \leq \inf \mathcal{F}\} \\ &= \sup\{\varphi(x) : \varphi \in \mathcal{F}_1\} \\ &= \underline{g}(x),\quad x \in \Omega \end{aligned}$$

Therefore

$$F(\underline{g}) = F(S(\underline{g})) = \inf \mathcal{F} \in \mathbb{H}(\Omega).$$

In the same way we prove that $F(\overline{g}) \in \mathbb{H}(\Omega)$. Then Theorem 24 implies that g is D-continuous. ■

Theorem 32 shows that the set of D-continuous functions contains all the interval hulls of all sets of usual real (point) valued continuous functions. However, in general, the set $\mathbb{G}_{hl}(\Omega)$ of the interval hulls of sets of continuous functions is only a subset of $\mathbb{G}(\Omega)$. More precisely, we have

$$\mathbb{G}_{hl}(\Omega) = \{f \in \mathbb{G}(\Omega) : \exists \varphi : \Omega \mapsto \overline{\mathbb{R}} : \varphi \text{ is continuous and } \varphi \subseteq f\}$$

Hence, $\mathbb{G}_{hl}(\Omega)$ essentially excludes the H-continuous functions. Indeed, the only H-continuous functions in the set $\mathbb{G}_{hl}(\Omega)$ are the continuous point valued functions on Ω. More precise analysis reveals that the set of all D-continuous functions functions actually consists of the interval hulls of the sets of H-continuous functions.

Appendix 1: Semi-continuous functions and Baire operators

We recall here the definitions of lower and upper semi-continuity as given in [4], which also include functions with extended real values.

Definition 33 *A function $f \in \mathcal{A}(\Omega)$ is called lower semi-continuous at $x \in \Omega$ if for every $m < f(x)$ there exists $\delta > 0$ such that $m < f(y)$ for all $y \in B_\delta(x)$. If $f(x) = -\infty$, then f is assumed lower semi-continuous at x.*

Definition 34 *A function $f \in \mathcal{A}(\Omega)$ is called upper semi-continuous at $x \in \Omega$ if for every $m > f(x)$ there exists $\delta > 0$ such that $m > f(y)$ for all $y \in B_\delta(x)$. If $f(x) = +\infty$, then f is assumed upper semi-continuous at x.*

Definition 35 *A function $f \in \mathcal{A}(\Omega)$ is called lower (upper) semi-continuous on Ω if it is lower (upper) semi-continuous at every point of Ω.*

The next theorem was proved in [4].

Theorem 36 *We have the following:*
a) Let $L \subseteq \mathcal{A}(\Omega)$ be a set of lower semi-continuous functions. Then function l defined by

$$l(x) = \sup\{f(x) : f \in L\}$$

is lower semi-continuous.
b) Let $U \subseteq \mathcal{A}(\Omega)$ be a set of upper semi-continuous functions. Then function u defined by

$$u(x) = \inf\{f(x) : f \in U\}$$

is upper semi-continuous.

From the definitions of the lower and upper Baire operators given in (6) and (7) and the above theorem one can immediately see that for every dense subset D of Ω and $f \in \mathbb{A}(D)$ we have

- $I(D, \Omega, f)$ is lower semi-continuous on Ω;
- $S(D, \Omega, f)$ is upper semi-continuous on Ω.

Furthermore, for a given dense subset D of Ω and $f \in \mathbb{A}(D)$ the functions $I(D, \Omega, f)$ and $S(D, \Omega, f)$ have the following optimality properties. For any $g \in \mathbb{A}(\Omega)$

$$\left.\begin{array}{c} g\text{- lower semi-continuous on } \Omega \\ g(x) \leq f(x),\ x \in D \end{array}\right\} \Longrightarrow g(x) \leq I(D, \Omega, f)(x),\ x \in \Omega;$$

$$\left.\begin{array}{c} g\text{- upper semi-continuous on } \Omega \\ g(x) \geq f(x),\ x \in D \end{array}\right\} \Longrightarrow g(x) \geq S(D, \Omega, f)(x),\ x \in \Omega.$$

Due to the above properties the functions $I(D, \Omega, f)$ and $S(D, \Omega, f)$ are also called respectively lower and upper semi-continuous envelops of the function f, see [5].

The following two concepts were introduced by Dilworth, [7].

Definition 37 *A function $f \in \mathcal{A}(\Omega)$ is called normal lower semi-continuous on Ω if it is lower semi-continuous and*

$$I(S(f)) = f\ .$$

Definition 38 *A function $f \in \mathcal{A}(\Omega)$ is called upper semi-continuous at if it is upper semi-continuous and*

$$S(I(f)) = f\ .$$

An important property of the normal lower and upper semi-continuous is that they can be represented through Dedekind cuts of the set of continuous functions. More precisely for a normal lower semi-continuous f we have

$$f(x) = \sup\{\varphi(x) : \varphi \in \mathcal{F}\},\ x \in \Omega,$$

where

$$\begin{aligned}
\mathcal{F} &= \{\varphi : \Omega \mapsto \overline{\mathbb{R}} : \varphi \text{ is continuous on } \Omega \text{ and } \varphi \leq \phi,\ \forall \phi \in \mathcal{G}\} \\
\mathcal{G} &= \{\phi : \Omega \mapsto \overline{\mathbb{R}} : \phi \text{ is continuous on } \Omega \text{ and } \phi \geq f\}.
\end{aligned}$$

In a similar way if f is normal upper semi-continuous then

$$f(x) = \inf\{\varphi(x) : \varphi \in \mathcal{F}\},\ x \in \Omega,$$

where

$$\begin{aligned}
\mathcal{F} &= \{\varphi : \Omega \mapsto \overline{\mathbb{R}} : \varphi \text{ is continuous on } \Omega \text{ and } \varphi \geq \phi,\ \forall \phi \in \mathcal{G}\} \\
\mathcal{G} &= \{\phi : \Omega \mapsto \overline{\mathbb{R}} : \phi \text{ is continuous on } \Omega \text{ and } \phi \leq f\}.
\end{aligned}$$

Appendix 2: Partial orders for intervals and interval functions

Several partial orders have historically been associated with the set $\mathbb{I}\overline{\mathbb{R}}$, namely,
(i) the inclusion relation $[\underline{a}, \overline{a}] \subseteq [\underline{b}, \overline{b}] \Longleftrightarrow \underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}$
(ii) the "strong" partial order $[\underline{a}, \overline{a}] \preceq [\underline{b}, \overline{b}] \Longleftrightarrow \overline{a} \leq \underline{b}$
(iii) the partial order defined by (3).
The use of the inclusion relation on the set $\mathbb{I}\overline{\mathbb{R}}$ is motivated by the applications of interval analysis to generating enclosures of solution sets. However, the role of partial orders extending the total order on $\overline{\mathbb{R}}$ has also been recognized in computing, see [6]. Both orders (ii) and (iii) are extensions of the order on $\overline{\mathbb{R}}$. The use of the order (ii) is based on the view point that inequality between intervals should imply inequality between their interiors. This approach is rather limiting since the order (ii) does not retain some essential properties of the order on $\overline{\mathbb{R}}$. For instance, a nondegenerate interval A and the interval $A + \varepsilon$ are not comparable with respect to the order (ii) when the positive real number ε is small enough. The partial order (iii) is introduced and studied by Markov, see [12], [11]. The results reported in this paper indicate that indeed the partial order (4) induced point-wise by (3) is an appropriate partial order to be associated with the considered spaces of interval functions. The partial order (5) induced point-wise by the

inclusion relation (i) also plays important role in the spaces where the functions are not 'thin' in the sense that they assume proper interval values on open subsets of the domain, e.g. the D-continuous or the S-continuous functions. The following concepts of order completeness and Dedekind order completeness were discussed in connection with both orders, namely (4) and (5).

Definition 39 *A partially ordered set P is called order complete if every nonempty subset A of P has both a supremum in P and an infimum in P.*

Definition 40 *A partially ordered set P is called Dedekind order complete if every nonempty subset A of P which is bounded from above has a supremum in P and every nonempty subset B of P which is bounded from below has an infimum in P.*

Definition 41 *Let P be a partially ordered set. A partially ordered set $P^{\#}$ is called a (Dedekind) order completion of P if*
i) $P^{\#}$ is (Dedekind) order complete;
ii) there exists an order isomorphism $\Phi : P \to \Phi(P) \subseteq P^{\#}$;
iii) if Q is (Dedekind) order complete and $\Phi(P) \subseteq Q \subseteq P^{\#}$ then $Q = P^{\#}$.

Clearly, a partially ordered set may be order complete only if all its subsets are bounded. Let us note here that all subsets of $\mathbb{H}(\Omega)$ are bounded with respect to the relation $\leq$. For example $v(x) = -\infty$, $x \in \Omega$, and $u(x) = +\infty$, $x \in \Omega$, are, respectively, lower and upper bounds of every subset of $\mathbb{H}(\Omega)$. However, this is not a property which is necessarily inherited by the subsets of $\mathbb{H}(\Omega)$, more precisely, if $\mathcal{B} \subseteq \mathbb{H}(\Omega)$ then the subsets of $\mathcal{B}$ are not necessarily bounded in $\mathcal{B}$.

One should note that the set of Hausdorff continuous function is Dedekind order complete with respect to the order relation inclusion ($\subseteq$). However, this statement hardly contains any information as the only bounded sets with respect to inclusions are the single function sets. Trivially each of these sets has both infimum and supremum equal to the function in the set. Hence we do not have any interest in inclusion within the space of the Hausdorff continuous functions. As shown in Section 7 the inclusion is an interesting partial order in the wider space of D-continuous functions.

Appendix 3: The set of nearly finite H-continuous functions

With every function $f \in \mathbb{A}(\Omega)$ we associate the set

$$\Gamma_{nf}(f) = \{x \in \Omega : +\infty \in f(x) \;\text{ or }\; -\infty \in f(x)\}. \tag{40}$$

Then,

$$f \text{ is nearly finite } \Longleftrightarrow \Gamma_{nf}(f) \text{ is closed and nowhere dense in } \Omega. \tag{41}$$

The condition in Definition 4 *simplifies* in the case of H-continuous function as follows.

Theorem 42 *For an H-continuous function f to be nearly finite it is sufficient to have finite values on a dense subset of Ω which need not be open as well.*

Proof. Let us assume that the function f assumes finite values on a set D, which is a dense subset of Ω. According to (41) we need to prove that $\Gamma_{nf}(f)$ is closed and nowhere dense in Ω.

Assume that $\Gamma_{nf}(f)$ is not nowhere dense. Let $\Gamma_{+\infty}(f) = \{x \in \Omega : +\infty \in f(x)\}$ and $\Gamma_{-\infty}(f) = \{x \in \Omega : -\infty \in f(x)\}$. Clearly $\Gamma_{nf}(f) = \Gamma_{+\infty}(f) \cup \Gamma_{-\infty}(f)$. Therefore at least one of the sets $\Gamma_{-\infty}(f)$ or $\Gamma_{+\infty}(f)$ is not nowhere dense, because the union of two nowhere dense sets is also nowhere dense. Let $\Gamma_{+\infty}(f)$ be not nowhere dense. Then, there exists an open set $G \subseteq X$ such that $\Gamma_{+\infty}(f) \cap G$ is dense in G. The function f assumes finite values on the set D which is dense in Ω. Hence $G \cap D \neq \emptyset$. Let $x \in G \cap D$. Using that $\Gamma_{+\infty}(f) \cap G$ is dense in G we obtain that for every $\delta > 0$ the intersection $B_\delta(x) \cap \Gamma_{+\infty}(f)$ is not empty. This implies

$$\sup\{z \in f(y) : y \in B_\delta(x)\} = +\infty$$

for every $\delta > 0$. Therefore

$$S(f)(x) = \inf_{\delta > 0} \sup\{z \in f(y) : y \in B_\delta(x)\} = +\infty.$$

On the other hand, since $x \in D$ we have $|f(x)| < +\infty$. The obtained contradiction shows that the assumption is false, i.e. $\Gamma_{nf}(f)$ is nowhere dense in Ω.

We will prove that $\Gamma_{nf}(f)$ is closed by proving that $\Omega \backslash \Gamma_{nf}(f)$ is open. Let $a \in \Omega \backslash \Gamma_{nf}(f)$. Then $I(f)(a) \in \mathbb{R}$ and $S(f)(a) \in \mathbb{R}$. Since $I(f)$ and $S(f)$ are lower and upper semi-continuous functions, respectively, there exists $\delta > 0$ such that for every $x \in B_\delta(a)$ we have

$$I(f)(a) - 1 \leq I(f)(x) \leq f(x) \leq S(f)(x) \leq S(f)(a) + 1.$$

Therefore $f(x)$ is finite for all $x \in B_\delta(a)$. Hence

$$a \in B_\delta(a) \subseteq \Omega \backslash \Gamma_{nf}(f).$$

Thus, $\Omega \backslash \Gamma_{nf}(f)$ is an open set. ■

The result in Theorem 42 is further detailed in its consequences as follows.

Theorem 43 *An H-continuous function f which is nearly finite has the additional property that its values are finite real numbers, that is, finite point intervals, everywhere on Ω, except for a set of first Baire category.*

Proof. An H-continuous function f assumes proper interval values only on the set W_f, given in (26), which is of first Baire category, i.e. it is a countable union of closed, nowhere dense sets. On the other side, the function f assumes nonfinite values only on the set $\Gamma_{nf}(f)$ defined through (40), which is closed and nowhere dense, see (41). Therefore, the function f assumes finite real values on the set $\Omega \backslash (W_f \cup \Gamma_{nf}(f))$. The set $W_f \cup \Gamma_{nf}(f)$ is of first Baire category, because it is a union of countably many closed, nowhere dense sets. This completes the proof. ■

We consider the set $\mathbb{H}_{nf}(\Omega)$ of all H-continuous nearly finite functions defined on Ω. The following theorems show that the set $\mathbb{H}_{nf}(\Omega)$ is Dedekind order complete with respect to the partial order defined by (4). Here we should note that a subset $\mathcal{F}$ of $\mathbb{H}_{nf}(\Omega)$ which is bounded from above or below may still contain functions with values $+\infty$ or $-\infty$ or unbounded closed intervals, this being compatible with the partial order defined by (4).

Theorem 44 *The set $\mathbb{H}_{nf}(\Omega)$ is Dedekind order complete, that is,*
a) if $\mathcal{F}$ is a nonempty subset of $\mathbb{H}_{nf}(\Omega)$ which is bounded from above then there exists $u \in \mathbb{H}_{nf}(\Omega)$ such that $u = \sup \mathcal{F}$;
b) if $\mathcal{F}$ is a nonempty subset of $\mathbb{H}_{nf}(\Omega)$ which is bounded from below then there exists $v \in \mathbb{H}_{nf}(\Omega)$ such that $v = \inf \mathcal{F}$.

Proof. We will prove only point a). Point b) is proved in a similar way. Since $\mathcal{F}$ is also a subset of the order complete space $\mathbb{H}(\Omega)$ the function $u = \sup \mathcal{F} \in \mathbb{H}(\Omega)$ is well defined. We only need to show that $u \in \mathbb{H}_{nf}(\Omega)$. We have already that u is H-continuous. Now we will show that it is nearly finite. Let $p \in \mathbb{H}_{nf}(\Omega)$ be an upper bound of $\mathcal{F}$. Clearly $u \leq p$. Let $f \in \mathcal{F}$. Since u is an upper bound of $\mathcal{F}$ we have

$$f(x) \leq u(x) \leq p(x),\ x \in \Omega.$$

Therefore u assumes finite values at all points of the set $\Omega \backslash (\Gamma_{nf}(f) \cup \Gamma_{nf}(u))$, which is open and dense in Ω. Hence $u \in \mathbb{H}_{nf}(\Omega)$. ■

References

[1] R Anguelov, Dedekind Order Completion of C(X) by Hausdorff Continuous Functions, Quaestiones Mathematicae, to appear.

[2] R. Anguelov and S. Markov, Extended segment analysis, *Freiburger Intervall - Berichte* 10 (1981), 1 - 63.

[3] R. Anguelov, E.E. Rosinger, Solution of Nonlinear PDEs by Hausdorff Continuous Functions (to appear).

[4] R. Baire, Lecons sur les Fonctions Discontinues, Collection Borel, Paris, 1905.

[5] M Bardi, I Capuzzo-Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkhäuser, Boston, Basel, Berlin, 1997.

[6] G. Birkhoff, The Role of Order in Computing, in *Reliability in Computing* (ed. R. Moore) (Academic Press, 1988), 357–378.

[7] R. P. Dilworth, The normal completion of the lattice of continuous functions, *Trans. Amer. Math. Soc.* **68** (1950), 427–438.

[8] W.A.J. Luxemburg, A.C. Zaanen, Riesz Spaces I, North Holland, Amsterdam, 1971.

[9] J. E. Mack and D. G. Johnson, The Dedekind completion of C(X), *Pacif. J. Math.* **20** (1967), 231-243.

[10] S. Markov, A nonstandard subtraction of intervals, *Serdica* **3** (1977), 359–370.

[11] S. Markov, Calculus for interval functions of a real variable, *Computing* **22** (1979), 325–337.

[12] S. Markov, Extended interval arithmetic involving infinite intervals, *Mathematica Balkanica* **6** (1992), 269–304.

[13] M.B. Oberguggenberger, E.E. Rosinger, Solution on Continuous Nonlinear PDEs through Order Completion, North-Holland, Amsterdam, London, New York, Tokyo, 1994.

[14] B. Sendov, Approximation of functions by algebraic polinomials with respect to a metric of Hausdorff type, *Annals of Sofia University, Mathematics* **55** (1962), 1-39.

[15] B. Sendov, *Hausdorff Approximations*, Kluwer Academic, Boston, 1990.

[16] R. Schneider, *Convex bodies: The Brunn - Minkowski theory*, Cambridge University Press, 1993.